ON THE KINETIC FOCI OF A CONSERVATIVE SYSTEM FOR ISOENERGETIC TRAJECTORIES

(O KINETICHESKIKH FOKUSAKH KONSERVATIVNOI SISTEMY DLIA ISOENERGETICHESKIKH TRAEKTORII)

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The question as to the character of the Lagrange action extremum and the related problem of kinetic foci for isoenergetic trajectories are dealt with in the book by Thomson and Tait [1], Bobylev's paper [2], and Suslov's book [3]. Thomson applied the theory of kinetic foci to the study of the orbital stability of the specified motion of a conservative system. He was followed in this by Routh and Zhukovskii.

The familiar method of determining kinetic foci for isoenergetic trajectories [2] involves expressing the generalized coordinates $q_2, \ldots, q_n$ of the system in terms of the coordinate q_1 from the differential equations of the trajectories in Jacobian form,

$$\frac{d}{dq_1}\frac{\partial \sqrt{R}}{\partial q_i'} - \frac{\partial \sqrt{R}}{\partial q_i} = 0 \qquad (i = 2, \ldots, n) \tag{1}$$

where the function R is given by

$$R = 2(h - \Pi)\sum_{s=1}^{n}\sum_{k=1}^{n} a_{sk} q_s' q_k'$$

(the primes denote differentiation with respect to q_1). Here Π and h are the potential energy and the total energy of the system, respectively (h = const) ; a_{sk} are the coefficients of the quadratic form which represents the kinetic energy T in the case of a conservative system. Realization of the known method requires one to solve system of differential equations (1) in the form

$$q_i = f_i(q_1, h, c_3, \ldots, c_{2n}) \qquad (i = 2, \ldots, n) \tag{2}$$

Here $c_3, \ldots, c_{2n}$ are arbitrary constants. The next step is to solve Equation

$$\begin{vmatrix} (\partial f_2/\partial c_3)_0 & (\partial f_2/\partial c_4)_0 & \ldots & (\partial f_2/\partial c_{2n})_0 \\ \ldots & \ldots & \ldots & \ldots \\ (\partial f_n/\partial c_3)_0 & (\partial f_n/\partial c_4)_0 & \ldots & (\partial f_n/\partial c_{2n})_0 \\ (\partial f_2/\partial c_3)_1 & (\partial f_2/\partial c_4)_1 & \ldots & (\partial f_2/\partial c_{2n})_1 \\ \ldots & \ldots & \ldots & \ldots \\ (\partial f_n/\partial c_3)_1 & (\partial f_n/\partial c_4)_1 & \ldots & (\partial f_n/\partial c_{2n})_1 \end{vmatrix} = 0 \tag{3}$$

for $q_1^{(1)}$. In this equation the subscript 0 denotes the initial position of the system, and the subscript 1 its position which is the kinetic focus conjugate to the initial position. However, functions of the form (2) can be obtained from trajectory equations (1) only in the simplest cases (due to the difficulty of integrating Equations (1)). This limits the applicability of the familiar method (*).

We shall present a method for determining the kinetic foci for isoenergetic systems which is based on the direct use of the equations of motion of a conservative system (**).

Let the totality of functions

$$q_i = q_i\ (t,\ c_1,\ .\ .\ .\ c_{2n}) \qquad (i = 1,\ .\ .\ .\ ,n) \tag{4}$$

be the general solution of the system of differential equations of motion of a conservative system (we shall consider the most general case, i.e. that in which the solution contains $2n$ constants but does not represent a Cauchy integral).

For fixed values of the constants $c_1, \ldots, c_{2n}$, Equations (4) define some true path of the system. Supplementing the constants with the infinitely small increments (variations) δc_k, we obtain the path

$$q_i^* = q_i(t,\ c_1 + \delta c_1,\ .\ .\ .\ .\ c_{2n} + \delta c_{2n}) \qquad (i = 1,\ .\ .\ .\ ,\ n) \tag{5}$$

which is infinitely close to the path (4) and is also a true path. The variations in the generalized coordinates upon transition from path (4) to path (5) are

$$\delta q_i = \sum_{k=1}^{2n} \frac{\partial q_i}{\partial c_k} \delta c_k \qquad (i = 1,\ .\ .\ .\ ,\ n) \tag{6}$$

Let paths (4) and (5) intersect at some position M_0 at the instant $t = t_0$. Then the variations δc_k must satisfy the conditions

$$\sum_{k=1}^{2n} \left(\frac{\partial q_i}{\partial c_k}\right)_{t=t_0} \delta c_k = 0 \qquad (i = 1,\ .\ .\ .\ ,\ n) \tag{7}$$

Eliminating the time t from the equations of motion (4), we obtain the equations of the system trajectory. We express t in terms of the coordinate q_1 from the first equation of system (4),

$$t = \tau\ (q_1,\ c_1,\ .\ .\ .\ ,\ c_{2n}) \tag{8}$$

Upon substitution from (8) into the other $(n - 1)$ equations of (4), we obtain the equations of the trajectory which corresponds to path (4),

$$q_i = q_i[\tau(q_1,\ c_1,\ .\ .\ .\ ,\ c_{2n}),\ c_1,\ .\ .\ .\ ,\ c_{2n}] \equiv \varphi_i(q_1,\ c_1,\ .\ .\ .\ ,\ c_{2n}) \quad (i = 2,\ .\ .\ .\ ,n) \tag{9}$$

Eliminating the time t from the equations of motion for path (5) by a similar procedure, we obtain the equations of the trajectory corresponding to path (5)

$$q_i^* = \varphi_i(q_1,\ c_1 + \delta c_1,\ .\ .\ .\ ,\ c_{2n} + \delta c_{2n}) \qquad (i = 2,\ .\ .\ .\ ,\ n) \tag{10}$$

The variations in the coordinates upon transition from trajectory (9) to trajectory (10) are

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*) We note that solution (2) of the system of trajectory equations (1) as a rule cannot be found in cases where one knows the equations of motion $q_i = q_i(t,\ c_1,\ .\ .\ .,\ c_{2n})$ $(i = 1,\ .\ .\ .,\ n)$, obtained by integrating the differential equations of motion of the system (second-kind Lagrange equations). In these cases functions (2) might conceivably be obtained from the equations of motion by eliminating from them the time t and by expressing two of their constants (e.g. c_1 and c_2) in terms of the other constants and in term of h of the equations: $q_1^{(0)} = q_1\ (t_0,\ c_1,\ .\ .\ .\ ,\ c_{2n})$, $h = h\ (c_1,\ .\ .\ .\ ,\ c_{2n})$. However, the mathematical difficulties involved usually render this technique for obtaining functions (2) useless.

**) We note that the kinetic foci for simultaneous paths (i.e. paths of the type considered in the Hamilton principle) are determined directly from the equations of motion of the system.

$$\delta q_i = \sum_{k=1}^{2n} \frac{\partial \varphi_i}{\partial c_k} \delta c_k \qquad (i = 2, \ldots, n) \tag{11}$$

Since paths (4) and (5) intersect at $t = t_0$, the corresponding trajectories (9) and (10) also intersect at $q_1 = q_{10}$ in the position M_0 (here q_{10} is the value of the constant q_1 at $t = t_0$). We arrive at the system of equations

$$\sum_{k=1}^{2n} \left(\frac{\partial \varphi_i}{\partial c_k}\right)_{q_1=q_{10}} \delta c_k = 0 \qquad (i = 2, \ldots, n) \tag{12}$$

The system of $(n - 1)$ equations (12)and Equation

$$\sum_{k=1}^{2n} \left(\frac{\partial q_1}{\partial c_k}\right)_{t=t_0} \delta c_k = 0 \tag{13}$$

are equivalent in the sistem of n Equations (7), i.e. they are fulfilled for the same values of the variations δc_k.

Let us suppose that in addition to the position M_0, the infinitely close trajectories (9) and (10) intersect at yet another position M_1. If the same constant value of the total mechanical energy h is retained on both trajectories, then the positions M_0 and M_1 of the conservative system are called conjugate kinetic foci for isoenergetic trajectories.

Since the total mechanical energy h on initial trajectory (9) is a function of the constants

$$h = h(c_1, \ldots, c_{2n}) \tag{14}$$

it follows that trajectories (9) and (10) are isoenergetic provided that

$$\delta h = \sum_{k=1}^{2n} \frac{\partial h}{\partial c_k} \delta c_k = 0 \tag{15}$$

Trajectories (9) and (10) intersect in the position M_1 (for $q_1 = q_{11}$) if the conditions

$$\delta q_i^{(1)} = \sum_{k=1}^{2n} \left(\frac{\partial \varphi_i}{\partial c_k}\right)_{q_1=q_{11}} \delta c_k = 0 \qquad (i = 2, \ldots, n) \tag{16}$$

are fulfilled.

Thus, the two positions M_0 and M_1 are conjugate kinetic foci if there exist values of the variations $\delta c_1, \ldots, \delta c_{2n}$ which satisfy the homogeneous system of $2n$ linear equations (12),(13),(15) and (16).

Let us reduce the indicated system of $2n$ equations to a system of $(2n - 2)$ equations. We begin by eliminating the variations δc_1 and δc_2 from Equations (13) and (15),

$$\delta c_1 = \left[\frac{\partial h}{\partial c_2}\left(\frac{\partial q_1}{\partial c_1}\right)_{t=t_0} - \frac{\partial h}{\partial c_1}\left(\frac{\partial q_1}{\partial c_2}\right)_{t=t_0}\right]^{-1} \sum_{k=3}^{2n} \left[\frac{\partial h}{\partial c_k}\left(\frac{\partial q_1}{\partial c_2}\right)_{t=t_0} - \frac{\partial h}{\partial c_2}\left(\frac{\partial q_1}{\partial c_k}\right)_{t=t_0}\right] \delta c_k$$

$$\delta c_2 = \left[\frac{\partial h}{\partial c_2}\left(\frac{\partial q_1}{\partial c_1}\right)_{t=t_0} - \frac{\partial h}{\partial c_1}\left(\frac{\partial q_1}{\partial c_2}\right)_{t=t_0}\right]^{-1} \sum_{k=3}^{2n} \left[\frac{\partial h}{\partial c_k}\left(\frac{\partial q_1}{\partial c_1}\right)_{t=t_0} - \frac{\partial h}{\partial c_1}\left(\frac{\partial q_1}{\partial c_k}\right)_{t=t_0}\right] \delta c_k$$

Substituting these values of δc_1 and δc_2 into Equations (12) and (16), we carry out some transformations and arrive at the homogeneous system of $(2n - 2)$ linear equations

$$\sum_{k=3}^{2n} D_{ik}^{(0)} \delta c_k = 0, \qquad \sum_{k=3}^{2n} D_{ik}^{(1)} \delta c_k = 0 \qquad (i = 2, \ldots, n) \tag{17}$$

Here $D_{ik}^{(0)}$ and $D_{ik}^{(1)}$ are values of the Jacobian

$$D_{ik}=\begin{vmatrix}\partial\varphi_i/\partial c_k & \partial\varphi_i/\partial c_2 & \partial\varphi_i/\partial c_1\\ \partial h/\partial c_k & \partial h/\partial c_2 & \partial h/\partial c_1\\ (\partial q_1/\partial c_k)_{t=t_0} & (\partial q_1/\partial c_2)_{t=t_0} & (\partial q_1/\partial c_1)_{t=t_0}\end{vmatrix}\qquad \begin{pmatrix}i=2,\ldots,n\\ k=3,\ldots,2n\end{pmatrix} \tag{18}$$

for $q_1 = q_{10}$ and $q_1 = q_{11}$.

For further transformation of the Jacobian D_{ik} we make use of the identities

$$q_1\,[\tau\,(q_1, c_1, \ldots, c_{2n}), c_1, \ldots, c_{2n}] \equiv q_1, \tag{19}$$

$$q_i\,[\tau\,(q_1, c_1, \ldots, c_{2n}), c_1, \ldots, c_{2n}] \equiv \varphi_i\,(q_1, c_1, \ldots, c_{2n}) \qquad (i = 2, \ldots, n) \tag{20}$$

Differentiating the left- and right-hand sides of these identities with respect to an arbitrary constant c_k , we obtain

$$\frac{\partial q_1}{\partial c_k}+\frac{\partial q_1}{\partial \tau}\frac{\partial \tau}{\partial c_k}=0 \qquad (k=1,\ldots,2n) \tag{21}$$

$$\frac{\partial q_i}{\partial c_k}+\frac{\partial q_i}{\partial \tau}\frac{\partial \tau}{\partial c_k}=\frac{\partial \varphi_i}{\partial c_k} \qquad (i=2,\ldots,n;\ k=1,\ldots,2n) \tag{22}$$

From Equations (21) we determine $\partial\tau/\partial c_k$ and then substitute these quantities into Equations (22). We then have

$$\frac{\partial \varphi_i}{\partial c_k}=\frac{\partial q_i}{\partial c_k}-\frac{\partial q_1}{\partial c_k}\frac{\partial q_i}{\partial \tau}\left(\frac{\partial q_1}{\partial \tau}\right)^{-1} \qquad (i=2,\ldots,n;\ k=1,\ldots,2n) \tag{23}$$

If we now replace $\partial\varphi_i/\partial c_k$ in Expression (18) for the Jacobian D_{ik} on the basis of Equations (23), converting in them from independent variable q_1 to the independent variable t by replacing τ by t , and then substitute the expression D_{ik} thus transformed into Equations (17), the latter become

$$\sum_{k=3}^{2n} B_{ik}^{(0)}\delta c_k=0, \qquad \sum_{k=3}^{2n} B_{ik}^{(1)}\delta c_k=0 \qquad (i=2,\ldots,n) \tag{24}$$

where $B_{ik}^{(0)}$ and $B_{ik}^{(1)}$ are the values of the Jacobian

$$B_{ik}=\begin{vmatrix}\partial q_i/\partial c_1 & \partial q_i/\partial c_2 & \partial q_i/\partial c_k & \partial q_i/\partial t\\ \partial q_1/\partial c_1 & \partial q_1/\partial c_2 & \partial q_1/\partial c_k & \partial q_1/\partial t\\ (\partial q_1/\partial c_1)_{t=t_0} & (\partial q_1/\partial c_2)_{t=t_0} & (\partial q_1/\partial c_k)_{t=t_0} & 0\\ \partial h/\partial c_1 & \partial h/\partial c_2 & \partial h/\partial c_k & 0\end{vmatrix}\qquad \begin{pmatrix}i=2,\ldots,n\\ k=3,\ldots,2n\end{pmatrix} \tag{25}$$

for $t = t_0$ and $t = t_1$. (We note that t_1 is the instant at which the system attains the kinetic focus M_1, proceeding along the initial trajectory (9); the time of the travel from M_0 to M_1 along trajectory (10) may, generally speaking, not coincide with the time of motion along the initial trajectory).

The system of homogeneous equations (24) admits of a nontivial solution for $\delta c_3, \ldots, \delta c_{2n}$ if its determinant vanishes, i.e. if

$$\Delta(t_1, t_0)=\begin{vmatrix}B_{2,3}^{(0)} & B_{2,4}^{(0)} & \ldots & B_{2,2n}^{(0)}\\ \cdot & \cdot & \cdot & \cdot\\ B_{n,3}^{(0)} & B_{n,4}^{(0)} & \ldots & B_{n,2n}^{(0)}\\ B_{2,3}^{(1)} & B_{2,4}^{(1)} & \ldots & B_{2,2n}^{(1)}\\ \cdot & \cdot & \cdot & \cdot\\ B_{n,3}^{(1)} & E_{n,4}^{(1)} & \ldots & B_{n,2n}^{(1)}\end{vmatrix}=0 \tag{26}$$

Having determined the root t_1 of Equation (26) which is closest to

t_0 $(t_1 > t_0)$, we obtain the instant at which the kinetic focus (*) conjugate to the position of the conservative system at $t = t_0$ for isoenergetic trajectories is attained. To determine the coordinates of the kinetic focus, one must substitute the resulting value of t_1 into the equations of motion of system (4).

E x a m p l e 1 . A material point of mass $m = 1$ is moving in a constant gravitational field. The equations of the point's motion are

$$q_1 = c_1 + c_2 t, \qquad q_2 = c_3 + c_4 t - {}^1/_2\, g t^2 \tag{27}$$

The total mechanical energy is

$$h = {}^1/_2\, (c_2^2 + c_4^2 + 2 g c_3) \tag{28}$$

Let us find the kinetic focus conjugate to the initial position of the point at the instant $t = t_0$. Equation (26) in this case becomes

$$\begin{vmatrix} B_{23}^{(0)} & B_{24}^{(0)} \\ B_{23}^{(1)} & B_{24}^{(1)} \end{vmatrix} = 0, \qquad \begin{array}{l} B_{23}^{(0)} = c_2^2, \\ B_{24}^{(0)} = c_2^2 t_0, \end{array} \qquad \begin{array}{l} B_{23}^{(1)} = c_2^2 + g\,(c_4 - g t_1)(t_1 - t_0) \\ B_{24}^{(1)} = c_2^2 t_1 + c_4\,(c_4 - g t_1)\,(t_1 - t_0) \end{array} \tag{29}$$

Since

$$B_{23} = \begin{vmatrix} 0 & 0 & 1 & (c_4 - gt) \\ 1 & t & 0 & c_2 \\ 1 & t_0 & 0 & 0 \\ 0 & c_2 & g & 0 \end{vmatrix} \qquad B_{24} = \begin{vmatrix} 0 & 0 & t & (c_4 - gt) \\ 1 & t & 0 & c_2 \\ 1 & t_0 & 0 & 0 \\ 0 & c_2 & c_4 & 0 \end{vmatrix}$$

$$B_{23} = c_2^2 + g\,(c_4 - gt)(t - t_0), \qquad B_{24} = c_2^2 t + c_4 (c_4 - gt)(t - t_0)$$

Solving Equation (29) for t_1 , we obtain

$$t_1 = \frac{c_2^2 + c_4^2 - c_4 g t_0}{g\,(c_4 - g t_0)} \tag{30}$$

In substituting the resulting expression for t_1 into Equations (2.7), we determine the coordinates of the kinetic focus (for $t_0 = 0$)

$$q_{11} = c_1 + \frac{c_2\,(c_2^2 + c_4^2)}{g c_4}, \qquad q_{21} = c_3 + \frac{c_4^4 - c_2^4}{2 g c_4^2} \tag{31}$$

The same result can be obtained through the use of the familiar method (see Lur'e's book [4], p.729).

In the case where we know the the general solution of the system of differential equations of motion of the conservative system and where this solution represents a Cauchy integral,

$$q_i = q_i(t,\, q_{10}, \ldots, q_{n0},\, q_{10}^{\cdot}, \ldots, q_{n0}^{\cdot}) \qquad (i = 1, \ldots, n) \tag{32}$$

the instant $t = t_1$ at which the kinetic focus is attained is found as that root of the equation (cited without derivation)

$$\Delta\,(t, t_0) = \begin{vmatrix} A_{22} & A_{23} & \ldots & A_{2n} \\ A_{32} & A_{33} & \ldots & A_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n2} & A_{n3} & \ldots & A_{nn} \end{vmatrix} = 0 \tag{33}$$

$$A_{ik} = \begin{vmatrix} \partial q_1 / \partial q_{10}^{\cdot} & \partial h / \partial q_{10}^{\cdot} & \partial q_i / \partial q_{10}^{\cdot} \\ \partial q_1 / \partial t & 0 & \partial q_i / \partial t \\ \partial q_1 / \partial q_{k0}^{\cdot} & \partial h / \partial q_{k0}^{\cdot} & \partial q_i / \partial q_{k0}^{\cdot} \end{vmatrix} \qquad (i, k = 2, \ldots\, n) \tag{34}$$

which is closest to t_0

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*) If Equation (26) does not have a root $t_1 = t_0$, then a kinetic focus does not exist.

E x a m p l e 2 . Two material points of masses $m_1 = m_2 = 1$ connected by weightless rigid rod of length $l = 1$ are moving in a vertical plane in a constant gravitational field. The initial conditions are specified at $t = t_0 = 0$. We are to determine the kinetic focus conjugate to the initial position of the system.

For our generalized coordinates q_1, q_2, q_3 we take the two Cartesian coordinates of one of the points and the angle of rotation of the rod. The equations of motion of the system are of the form

$$\begin{aligned} q_1 &= -{}^1/_2 \cos (q_{30}^{\cdot} t + q_{30}) + (q_{10}^{\cdot} - {}^1/_2 q_{30}^{\cdot} \sin q_{30})\, t + (q_{10} + {}^1/_2 \cos q_{30}) \\ q_2 &= -{}^1/_2 \sin (q_{30}^{\cdot} t + q_{30}) - {}^1/_2 g t^2 + (q_{20}^{\cdot} + {}^1/_2\, q_{30}^{\cdot} \cos q_{30})\, t + (q_{20} + {}^1/_2 \sin q_{30}) \\ q_3 &= q_{30}^{\cdot}\, t + q_{30} \end{aligned} \qquad (35)$$

The total mechanical energy is

$$h = {}^1/_2 (q_{10}^{\cdot 2} + q_{20}^{\cdot 2}) + \frac{1}{2} [(q_{10}^{\cdot} - q_{30}^{\cdot} \sin q_{30})^2 + (q_{20}^{\cdot} + q_{30}^{\cdot} \cos q_{30})^2] + g (2 q_{20} + \sin q_{30})$$

Equation (33) in this case is of the form

$$\Delta (t, t_0) = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = 0 \qquad (36)$$

where A_{ik} $(i, k = 2, 3)$ is a determinant of the form (34). The root of Equation (36) is

$$t_1 = \frac{(q_{10}^{\cdot 2} + q_{20}^{\cdot 2}) + (q_{10}^{\cdot} - q_{30}^{\cdot} \sin q_{30})^2 + (q_{20}^{\cdot} + q_{30}^{\cdot} \cos q_{30})^2}{g\,(2 q_{20}^{\cdot} + q_{30}^{\cdot} \cos q_{30})} \qquad (37)$$

Substituting the resulting value $t = t_1$ into equation of motion (34), we obtain the coordinates of the required kinetic focus.

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